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General affine surface areas

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Abstract

Two families of general affine surface areas are introduced. Basic properties and affine isoperimetric inequalities for these new affine surface areas as well as for L_ϕ affine surface areas are established.

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Finding the *right* notion of affine surface area was one of the first questions asked within affine differential geometry. At the beginning of the last century, Blaschke [5] and his School studied this question and introduced equi-affine surface area – a notion of surface area that is equi-affine invariant, that is, $\mathrm{SL}(n)$ and translation invariant. The first fundamental result regarding equi-affine surface area was the classical affine isoperimetric inequality of differential geometry [5]. Numerous important results regarding equi-affine surface area were obtained in recent years (see, for example, [1,2,45,48–51]). Using valuations on convex bodies, the author and Reitzner [27] were able to characterize a much richer family of affine surface areas (see Theorem 2). Classical equi-affine and centro-affine surface areas as well as all L_p affine surface areas for $p > 0$ belong to this family of L_ϕ affine surface areas.

The present paper has two aims. The first is to establish affine isoperimetric inequalities and basic duality relations for all L_ϕ affine surface areas. The second aim is to define new general notions of affine surface area that complement L_ϕ affine surface areas and include L_p affine surface

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areas for $p < -n$ and $-n < p < 0$. Let $\mathcal{K}_0^n$ denote the space of convex bodies, that is, compact convex sets, in $\mathbb{R}^n$ that contain the origin in their interiors. Whereas L_ϕ affine surface areas are always finite and are upper semicontinuous functionals on $\mathcal{K}_0^n$, the affine surface areas of the new families are infinite for certain convex bodies including polytopes and are lower semicontinuous functionals on $\mathcal{K}_0^n$. Basic properties and affine isoperimetric inequalities for these new affine surface areas are established. In Section 6, it is conjectured that together with L_ϕ affine surface areas, these new affine surface areas constitute – in a certain sense – *all* affine surface areas.

For a smooth convex body $K \subset \mathbb{R}^n$, equi-affine surface area is defined by

$$\Omega(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{n+1}}\, d\mu_K(x). \tag{1}$$

Here $d\mu_K(x) = x \cdot u(K,x)\, d\mathcal{H}(x)$ is the cone measure on ∂K, $x \cdot u$ is the standard inner product of $x, u \in \mathbb{R}^n$, $u(K,x)$ is the exterior unit normal vector to K at $x \in \partial K$, $\mathcal{H}$ is the $(n-1)$-dimensional Hausdorff measure,

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{(x \cdot u(K,x))^{n+1}},$$

and $\kappa(K,x)$ is the Gaussian curvature of K at x. Note that $\kappa_0(K,x)$ is (up to a constant) just a power of the volume of the origin-centered ellipsoid osculating K at x and thus is an $\mathrm{SL}(n)$ covariant notion. Also μ_K is an $\mathrm{SL}(n)$ covariant notion. Thus Ω is easily seen to be $\mathrm{SL}(n)$ invariant and it is also easily seen to be translation invariant. The notion of equi-affine surface area is fundamental in affine differential and convex geometry. Since many basic problems in discrete and stochastic geometry are equi-affine invariant, equi-affine surface area has found numerous applications in these fields (see, for example, [3,4,12,40]).

The extension of the definition of equi-affine surface area to general convex bodies was obtained much more recently in a series of papers [21,29,43]. Since $\kappa_0(K,\cdot)$ exists μ_K a.e. on ∂K by Aleksandrov's differentiability theorem, definition (1) still can be used. The long conjectured upper semicontinuity of equi-affine surface area (for smooth surfaces as well as for general convex surfaces) was proved by Lutwak [29] in 1991, that is,

$$\limsup_{j\to\infty} \Omega(K_j) \leqslant \Omega(K)$$

for any sequence of convex bodies K_j converging to K (in the Hausdorff metric). Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$. Schütt [42] showed that Ω is a valuation on $\mathcal{K}^n$, that is,

$$\Omega(K) + \Omega(L) = \Omega(K \cup L) + \Omega(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. An equi-affine version of Hadwiger's celebrated classification theorem [18] was established in [26]: (up to multiplication with a non-negative constant) equi-affine surface area is the unique upper semicontinuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathcal{K}^n$ that vanishes on polytopes.

During the past decade and a half, there has been an explosive growth of an L_p extension of the classical Brunn–Minkowski theory (see, for example, [6–8,15–17,24,25,31,34–38,46,47]). Within this theory, L_p affine surface area is the notion corresponding to equi-affine surface area in the classical Brunn–Minkowski theory. For $p > 1$, L_p affine surface area, Ω_p, was introduced by Lutwak [32] and shown to be $\mathrm{SL}(n)$ invariant, homogeneous of degree $q = p(n-p)/(n+p)$ (that is, $\Omega_p(tK) = t^q \Omega_p(K)$ for $t > 0$), and upper semicontinuous on $\mathcal{K}_0^n$. Hug [19] defined L_p affine surface area for every $p > 0$ and obtained the following representation for $K \in \mathcal{K}_0^n$:

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}}\, d\mu_K(x). \tag{2}$$

Note that $\Omega_1 = \Omega$ and that Ω_n is the classical (and $\mathrm{GL}(n)$ invariant) centro-affine surface area. Geometric interpretations of L_p affine surface areas were obtained in [11,39,44,52], and an application of L_p affine surface areas to partial differential equations is given in [33].

The L_p affine surface areas for $p > 0$ are special cases of the following family of affine surface areas introduced in [27]. Let $\mathrm{Conc}(0,\infty)$ be the set of functions $\phi : (0,\infty) \to (0,\infty)$ such that ϕ is concave, $\lim_{t\to 0} \phi(t) = 0$, and $\lim_{t\to\infty} \phi(t)/t = 0$. Set $\phi(0) = 0$. For $\phi \in \mathrm{Conc}(0,\infty)$, we define the L_ϕ affine surface area of K by

$$\Omega_\phi(K) = \int_{\partial K} \phi\big(\kappa_0(K,x)\big)\, d\mu_K(x). \tag{3}$$

The following basic properties of L_ϕ affine surface areas were established in [27]. Let $\mathcal{P}_0^n$ denote the set of convex polytopes containing the origin in their interiors.

Theorem 1. *([27].) If $\phi \in \mathrm{Conc}(0,\infty)$, then $\Omega_\phi(K)$ is finite for every $K \in \mathcal{K}_0^n$ and $\Omega_\phi(P) = 0$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_\phi : \mathcal{K}_0^n \to [0,\infty)$ is both upper semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

The family of L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0,\infty)$ is distinguished by the following basic properties (see [23] and [27], for characterizations of functionals that do not necessarily vanish on polytopes).

Theorem 2. *([27].) If $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$, then there exists $\phi \in \mathrm{Conc}(0,\infty)$ such that*

$$\Phi(K) = \Omega_\phi(K)$$

for every $K \in \mathcal{K}_0^n$.

One of the most important inequalities of affine geometry is the classical affine isoperimetric inequality. The following theorem establishes affine isoperimetric inequalities for all L_ϕ affine surface areas. Let $\mathcal{K}_c^n$ denote the space of $K \in \mathcal{K}_0^n$ that have their centroids at the origin and $|K|$ the n-dimensional volume of K.

Theorem 3. *Let $K \in \mathcal{K}_c^n$ and $B_K \in \mathcal{K}_c^n$ be the ball such that $|B_K| = |K|$. If $\phi \in \mathrm{Conc}(0,\infty)$, then*

$$\Omega_\phi(K) \leqslant \Omega_\phi(B_K)$$

and there is equality for strictly increasing ϕ if and only if K is an ellipsoid.

For $\phi(t) = t^{1/(n+1)}$ and smooth convex bodies, Theorem 3 is the classical affine isoperimetric inequality of differential geometry. For general convex bodies, proofs of the classical affine isoperimetric inequality were given by Leichtweiß [21], Lutwak [29], and Hug [19]. For L_p affine surface areas, the affine isoperimetric inequality was established by Lutwak [32] for $p > 1$ and by Werner and Ye [53] for $p > 0$.

Polarity on convex bodies induces the following duality on L_ϕ affine surface areas. Let $K^* = \{x \in \mathbb{R}^n\colon\ x \cdot y \leqslant 1 \text{ for } y \in K\}$ denote the polar body of $K \in \mathcal{K}_0^n$. For $\phi \in \mathrm{Conc}(0,\infty)$, define $\phi_* : (0,\infty) \to (0,\infty)$ by $\phi_*(s) = s\phi(1/s)$.

Theorem 4. *If $\phi \in \mathrm{Conc}(0,\infty)$, then $\Omega_\phi(K^*) = \Omega_{\phi_*}(K)$ holds for every $K \in \mathcal{K}_0^n$.*

For L_p affine surface areas and $p > 0$, Theorem 4 is due to Hug [20]: $\Omega_p(K^*) = \Omega_{n^2/p}(K)$ for every $K \in \mathcal{K}_0^n$.

An alternative definition of L_p affine surface area uses integrals of the curvature function $f(K,\cdot)$ over the unit sphere $\mathbb{S}^{n-1}$ (see [32]). This approach can also be used for L_ϕ affine surface areas.

Theorem 5. *If $\phi \in \mathrm{Conc}(0,\infty)$, then*

$$\Omega_\phi(K) = \int\limits_{\mathbb{S}^{n-1}} \phi_*\big(a_0(K,u)\big)\, d\nu_K(u)$$

for every $K \in \mathcal{K}_0^n$.

Here $a_0(K,u) = f_{-n}(K,u) = h(K,u)^{n+1} f(K,u)$ is the L_p curvature function of K (see [32]) for $p = -n$, while $h(K,u)$ is the support function of K, and $d\nu_K(u) = d\mathcal{H}(u)/h(K,u)^n$ (see Section 1 for precise definitions). For L_p affine surface areas and $p > 0$, Theorem 5 is due to Hug [19].

The family of L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0,\infty)$ includes all $\mathrm{SL}(n)$ invariant and upper semicontinuous valuations on $\mathcal{K}_0^n$ that vanish on polytopes and, in particular, all L_p affine surface areas for $p > 0$. However, L_p affine surface areas for $p < 0$ do not belong to the family of L_ϕ affine surface areas. Recent results by Meyer and Werner [39], Schütt and Werner [44], Werner [52], and Werner and Ye [53] underline the importance of L_p affine surface area also for $p < 0$.

A new family of affine surface areas generalizes L_p affine surface area for $-n < p < 0$. Let $\mathrm{Conv}(0,\infty)$ be the set of functions $\psi : (0,\infty) \to (0,\infty)$ such that ψ is convex, $\lim_{t\to 0}\psi(t) = \infty$, and $\lim_{t\to\infty}\psi(t) = 0$. Set $\psi(0) = \infty$. For $\psi \in \mathrm{Conv}(0,\infty)$, we define the L_ψ affine surface area of K by

$$\Omega_\psi(K) = \int\limits_{\partial K} \psi\big(\kappa_0(K,x)\big)\, d\mu_K(x). \tag{4}$$

The following theorem establishes basic properties of L_ψ affine surface areas.

Theorem 6. *If $\psi \in \mathrm{Conv}(0,\infty)$, then $\Omega_\psi(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_\psi(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_\psi : \mathcal{K}_0^n \to (0,\infty]$ is both lower semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

An immediate consequence of Theorem 6 is the following result for L_p affine surface area.

Corollary 7. *If $-n < p < 0$, then $\Omega_p(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_p(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_p : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

Affine isoperimetric inequalities for L_ψ affine surface areas are established in

Theorem 8. *Let $K \in \mathcal{K}_c^n$ and $B_K \in \mathcal{K}_c^n$ be the ball such that $|B_K| = |K|$. If $\psi \in \mathrm{Conv}(0, \infty)$, then*

$$\Omega_\psi(K) \geqslant \Omega_\psi(B_K)$$

and there is equality for strictly decreasing ψ if and only if K is an ellipsoid.

For $\psi(t) = t^{p/(n+p)}$ and $-n < p < 0$, this result was proved (in a different way) by Werner and Ye [53].

For $\psi \in \mathrm{Conv}(0, \infty)$, define $\Omega_\psi^* : \mathcal{K}_0^n \to (0, \infty]$ by $\Omega_\psi^*(K) := \Omega_\psi(K^*)$. The following theorem establishes basic properties of these affine surface areas.

Theorem 9. *If $\psi \in \mathrm{Conv}(0, \infty)$, then $\Omega_\psi^*(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_\psi^*(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_\psi^* : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

The family of affine surface areas Ω_ψ^* for $\psi \in \mathrm{Conv}(0, \infty)$ complements L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0, \infty)$ and L_ψ affine surface areas for $\psi \in \mathrm{Conv}(0, \infty)$. Whereas L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0, \infty)$ include affine surface areas homogeneous of degree q for all $|q| < n$ and L_ψ affine surface areas for $\psi \in \mathrm{Conv}(0, \infty)$ include affine surface areas homogeneous of degree q for all $q > n$, the new family includes affine surface areas homogeneous of degree q for all $q < -n$.

The next theorem gives a representation of Ω_ψ^* corresponding to that of Theorem 5.

Theorem 10. *If $\psi \in \mathrm{Conv}(0, \infty)$, then*

$$\Omega_\psi^*(K) = \int\limits_{\mathbb{S}^{n-1}} \psi\big(a_0(K, u)\big)\, d\nu_K(u)$$

for every $K \in \mathcal{K}_0^n$.

For $p < -n$, L_p affine surface area was defined by Schütt and Werner [44] using (2). Here a different approach is used and a different definition of L_p affine surface areas for $p < -n$ is given:

$$\Omega_p(K) := \int\limits_{\mathbb{S}^{n-1}} a_0(K, u)^{\frac{n}{n+p}}\, d\nu_K(u). \tag{5}$$

By Theorem 10, $\Omega_p(K) = \Omega_{n^2/p}^*(K) = \Omega_\psi^*(K)$ with $\psi(t) = t^{n/(n+p)}$ and $p < -n$.

An immediate consequence of Theorem 9 is the following result for L_p affine surface area as defined by (5).

Corollary 11. *If $p < -n$, then $\Omega_p(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_p(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_p : \mathcal{K}_0^n \to (0, \infty]$ is both lower semicontinuous and an* SL(n) *invariant valuation.*

1. Tools

Basic notions on convex bodies and their curvature measures are collected. For detailed information, see [10,13,41]. Let $K \in \mathcal{K}_0^n$. The support function of K is defined for $x \in \mathbb{R}^n$ by

$$h(K, x) = \max\{x \cdot y \colon\ y \in K\}.$$

The radial function of K is defined for $x \in \mathbb{R}^n$ and $x \neq 0$ by

$$\rho(K, x) = \max\{t > 0 \colon\ tx \in K\}.$$

Note that these definitions immediately imply that

$$\rho(K, x) = 1 \quad \text{for } x \in \partial K, \tag{6}$$

$$\rho(K, tu) = \frac{1}{t}\rho(K, x) \quad \text{for } t > 0, \tag{7}$$

and

$$h(K, u) = \frac{1}{\rho(K^*, u)}, \tag{8}$$

where K^* is the polar body of K.

Let $\mathcal{B}(\mathbb{R}^n)$ denote the family of Borel sets in $\mathbb{R}^n$ and $\sigma(K, \beta)$ the spherical image of $\beta \in \mathcal{B}(\mathbb{R}^n)$, that is, the set of all exterior unit normal vectors of K at points of β. Note that $\sigma(K, \beta)$ is Lebesgue measurable for each $\beta \in \mathcal{B}(\mathbb{R}^n)$. For a sequence of convex bodies $K_j \in \mathcal{K}_0^n$ converging to $K \in \mathcal{K}_0^n$ and a closed set $\beta \subset \mathbb{R}^n$, we have

$$\limsup_{j \to \infty} \sigma(K_j, \beta) \subset \sigma(K, \beta). \tag{9}$$

For $\beta \in \mathcal{B}(\mathbb{R}^n)$, set

$$C(K, \beta) = \int\limits_{\sigma(K,\beta)} \frac{d\mathcal{H}(u)}{h(K, u)^n},$$

where $\mathcal{H}$ denotes the $(n-1)$-dimensional Hausdorff measure. Hence $C(K, \cdot)$ is a Borel measure on $\mathbb{R}^n$ that is concentrated on ∂K. By (8), we obtain

$$C(K, \partial K) = n\left|K^*\right|. \tag{10}$$

It follows from (9) that for every closed set $\beta \subset \mathbb{R}^n$,

$$\limsup_{j\to\infty} C(K_j, \beta) \leqslant C(K, \beta). \tag{11}$$

Let $C_0(K, \cdot) : \mathcal{B}(\mathbb{R}^n) \to [0, \infty)$ be the 0-th curvature measure of the convex body K (see [41], Section 4.2). For $\beta \in \mathcal{B}(\mathbb{R}^n)$, we have

$$C_0(K, \beta) = \mathcal{H}\big(\sigma(K, \beta)\big). \tag{12}$$

We decompose the measure $C_0(K, \cdot)$ into measures absolutely continuous and singular with respect to $\mathcal{H}$, say, $C_0(K, \cdot) = C_0^a(K, \cdot) + C_0^s(K, \cdot)$. Note that

$$\frac{dC_0^a(K, \cdot)}{d\mathcal{H}} = \kappa(K, \cdot). \tag{13}$$

Let $\operatorname{reg} K$ denote the set of regular boundary points of K, that is, boundary points with a unique exterior unit normal vector. From (12), we obtain for $\omega \subset \operatorname{reg} K$ and $\omega \in \mathcal{B}(\mathbb{R}^n)$,

$$C(K, \omega) = \int\limits_{\sigma(K,\omega)} \frac{d\mathcal{H}(u)}{h(K, u)^n} = \int\limits_{\omega} \frac{dC_0(K, x)}{(x \cdot u(K, x))^n}. \tag{14}$$

We decompose the measure $C(K, \cdot)$ into measures absolutely continuous and singular with respect to the measure μ_K, say, $C(K, \cdot) = C^a(K, \cdot) + C^s(K, \cdot)$. The singular part is concentrated on a μ_K null set $\omega_0 \subset \partial K$, that is, for $\beta \in \mathcal{B}(\mathbb{R}^n)$

$$C^s(K, \beta \backslash \omega_0) = 0. \tag{15}$$

Since $C^a(K, \cdot)$ is concentrated on $\operatorname{reg} K$, (13) and (14) imply for $\omega \subset \partial K$ and $\omega \in \mathcal{B}(\mathbb{R}^n)$,

$$C^a(K, \omega) = \int\limits_{\omega} \frac{\kappa(K, x)}{(x \cdot u(K, x))^n}\, d\mathcal{H}(x) = \int\limits_{\omega} \kappa_0(K, x)\, d\mu_K(x). \tag{16}$$

Combined with (10), this implies

$$\int\limits_{\partial K} \kappa_0(K, x)\, d\mu_K(x) \leqslant n\big|K^*\big|. \tag{17}$$

Hug [20] proved that for almost all $x \in \partial K$,

$$\kappa(K, x) = \left(\frac{x}{|x|} \cdot u_K(x)\right)^{n+1} f\left(K^*, \frac{x}{|x|}\right).$$

Hence we have for almost all $y \in \partial K^*$,

$$\kappa_0\big(K^*, y\big) = a_0\left(K, \frac{y}{|y|}\right). \tag{18}$$

Here $|x|$ denotes the length of x.

2. Proof of Theorems 3 and 8

Let $\phi \in \mathrm{Conc}(0,\infty)$ and $K \in \mathcal{K}_c^n$. By definition (3), Jensen's inequality, (17), and the monotonicity of ϕ, we obtain

$$\begin{aligned}\Omega_\phi(K) &= \int_{\partial K} \phi\big(\kappa_0(K,x)\big)\,d\mu_K(x)\\ &\leqslant n|K|\phi\bigg(\frac{1}{n|K|}\int_{\partial K}\kappa_0(K,x)\,d\mu_K(x)\bigg)\\ &\leqslant n|K|\phi\bigg(\frac{|K^*|}{|K|}\bigg).\end{aligned}$$

For origin-centered ellipsoids, $\kappa_0(K,\cdot)$ is constant and there is equality in the above inequalities. Now we use the Blaschke–Santaló inequality: for $K \in \mathcal{K}_c^n$

$$|K||K^*| \leqslant \left|B^n\right|^2$$

with equality precisely for origin-centered ellipsoids (see, for example, [28]). Here B^n is the unit ball in $\mathbb{R}^n$. We obtain

$$\Omega_\phi(K) \leqslant n|K|\phi\bigg(\frac{|K^*|}{|K|}\bigg) \leqslant n|K|\phi\bigg(\frac{|B^n|^2}{|K|^2}\bigg) = \Omega_\phi(B_K). \tag{19}$$

For ϕ strictly increasing, equality in the second inequality of (19) holds if and only if there is equality in the Blaschke–Santaló inequality, that is, precisely for ellipsoids. This completes the proof of Theorem 3 and the proof of Theorem 8 follows along similar lines.

3. Proof of Theorems 4 and 9

Define Ω_ϕ^* on $\mathcal{K}_0^n$ by $\Omega_\phi^*(K) := \Omega_\phi(K^*)$. Since Ω_ϕ is upper semicontinuous, so is Ω_ϕ^*. For $K, L, K \cup L \in \mathcal{K}_0^n$, we have

$$(K\cup L)^* = K^* \cap L^* \quad \text{and} \quad (K \cap L)^* = K^* \cup L^*.$$

Since Ω_ϕ is a valuation, this implies that

$$\begin{aligned}\Omega_\phi^*(K) + \Omega_\phi^*(L) &= \Omega_\phi\big(K^*\big) + \Omega_\phi\big(L^*\big)\\ &= \Omega_\phi\big(K^*\cup L^*\big) + \Omega_\phi\big(K^*\cap L^*\big)\\ &= \Omega_\phi\big((K\cap L)^*\big) + \Omega_\phi\big((K\cup L)^*\big)\\ &= \Omega_\phi^*(K\cap L) + \Omega_\phi^*(K\cup L),\end{aligned}$$

that is, Ω_ϕ^* is a valuation on $\mathcal{K}_0^n$. For $A \in \mathrm{SL}(n)$ and $K \in \mathcal{K}_0^n$, we have $(AK)^* = A^{-t}K^*$, where A^{-t} denotes the inverse of the transpose of A. Since Ω_ϕ is $\mathrm{SL}(n)$ invariant, this implies $\Omega_\phi^*(AK) = \Omega_\phi^*(K)$, that is, $\Omega_\phi^* : \mathcal{K}_0^n \to \mathbb{R}$ is $\mathrm{SL}(n)$ invariant. Since Ω_ϕ vanishes on polytopes, so does Ω_ϕ^*. Therefore Ω_ϕ^* satisfies the assumptions of Theorem 2. Thus there exists $\alpha \in \mathrm{Conc}(0,\infty)$ such that $\Omega_\phi^* = \Omega_\alpha$. Let B^n denote the unit ball in $\mathbb{R}^n$. For $r > 0$, we obtain from (3) that

$$\Omega_\alpha\big(rB^n\big) = n\big|B^n\big|r^n\alpha\left(\frac{1}{r^{2n}}\right)$$

and

$$\Omega_\phi^*\big(rB^n\big) = \Omega_\phi\left(\frac{1}{r}B^n\right) = \frac{n|B^n|}{r^n}\phi\big(r^{2n}\big).$$

This shows that $\alpha = \phi_*$ and completes the proof of Theorem 4. The proof of Theorem 9 follows along the lines of the proof that Ω_ϕ^* satisfies the assumptions of Theorem 2.

4. Proof of Theorems 5 and 10

Define $y : \mathbb{S}^{n-1} \to \partial K^*$ by $u \mapsto \rho(K^*,u)u$. Note that this is a Lipschitz function. For the Jacobian Jy of y, we have a.e. on $\mathbb{S}^{n-1}$,

$$Jy(u) = \frac{\rho(K^*,u)^{n-1}}{u\cdot u_{K^*}(\rho(K^*,u)u)} \tag{20}$$

(see, for example, [20]). By the area formula (see, for example, [9]), we have for every a.e. defined function $g : \mathbb{S}^{n-1} \to [0,\infty]$,

$$\int\limits_{\mathbb{S}^{n-1}} g(u)Jy(u)\,d\mathcal{H}(u) = \int\limits_{\partial K^*} g\left(\frac{y}{|y|}\right)d\mathcal{H}(y).$$

Setting

$$g(u) = \frac{\tau(a_0(K,u))}{h(K,u)^n Jy(u)}$$

for $\tau : [0,\infty] \to [0,\infty]$, we get by (6), (7), (8), and (18),

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} \tau\big(a_0(K,u)\big)\,d\nu_K(u) &= \int\limits_{\mathbb{S}^{n-1}} \tau\big(a_0(K,u)\big)\frac{d\mathcal{H}(u)}{h(K,u)^n}\\ &= \int\limits_{\partial K^*} \tau\big(\kappa_0\big(K^*,y\big)\big)\frac{\frac{y}{|y|}\cdot u_{K^*}(y)}{\rho(K^*,\frac{y}{|y|})^{n-1}}\rho\left(K^*,\frac{y}{|y|}\right)^n d\mathcal{H}(y)\\ &= \int\limits_{\partial K^*} \tau\big(\kappa_0\big(K^*,y\big)\big)\,d\mu_{K^*}(y).\end{aligned}$$

For $\tau \in \mathrm{Conv}(0,\infty)$, this implies Theorem 10. To obtain Theorem 5, we set $\tau = \phi_* \in \mathrm{Conc}(0,\infty)$ and apply Theorem 4.

5. Proof of Theorem 6

Let $\psi \in \mathrm{Conv}(0,\infty)$ and $K \in \mathcal{K}_0^n$. Note that ψ is strictly decreasing and positive. By definition (4), the Jensen inequality, (17), and the monotonicity of ψ, we obtain

$$\begin{aligned}\Omega_\psi(K) &= \int_{\partial K} \psi\big(\kappa_0(K,x)\big)\, d\mu_K(x)\\ &\geqslant n|K|\psi\bigg(\frac{1}{n|K|}\int_{\partial K} \kappa_0(K,x)\, d\mu_K(x)\bigg)\\ &\geqslant n|K|\psi\bigg(\frac{|K^*|}{|K|}\bigg).\end{aligned}$$

This shows that $\Omega_\psi(K) > 0$. The $\mathrm{SL}(n)$ invariance of Ω_ψ follows immediately from the definition. So does the fact that $\Omega_\psi(P) = \infty$ for $P \in \mathcal{P}_0^n$.

Next, we show that Ω_ψ is a valuation on $\mathcal{K}_0^n$, that is, for $K, L \in \mathcal{K}_0^n$ such that $K \cup L \in \mathcal{K}_0^n$,

$$\Omega_\psi(K \cup L) + \Omega_\psi(K \cap L) = \Omega_\psi(K) + \Omega_\psi(L). \tag{21}$$

Let $K^c = \{x \in \mathbb{R}^n:\ x \notin K\}$ and let $\operatorname{int} K$ denote the interior of K. We follow Schütt [42] (see also [14]) and work with the decompositions

$$\begin{aligned}\partial(K \cup L) &= (\partial K \cap \partial L) \cup \big(\partial K \cap L^c\big) \cup \big(\partial L \cap K^c\big),\\ \partial(K \cap L) &= (\partial K \cap \partial L) \cup (\partial K \cap \operatorname{int} L) \cup (\partial L \cap \operatorname{int} K),\\ \partial K &= (\partial K \cap \partial L) \cup \big(\partial K \cap L^c\big) \cup (\partial K \cap \operatorname{int} L),\\ \partial L &= (\partial K \cap \partial L) \cup \big(\partial L \cap K^c\big) \cup (\partial L \cap \operatorname{int} K),\end{aligned}$$

where all unions on the right-hand side are disjoint. Note that for x such that the curvatures $\kappa_0(K,x)$, $\kappa_0(L,x)$, $\kappa_0(K \cup L,x)$, and $\kappa_0(K \cap L,x)$ exist,

$$u(K,x) = u(L,x) = u(K \cup L,x) = u(K \cap L,x) \tag{22}$$

and

$$\begin{aligned}\kappa_0(K \cup L,x) &= \min\big\{\kappa_0(K,x), \kappa_0(L,x)\big\},\\ \kappa_0(K \cap L,x) &= \max\big\{\kappa_0(K,x), \kappa_0(L,x)\big\}.\end{aligned} \tag{23}$$

To prove (21), we use (4), split the involved integrals using the above decompositions, and use (22) and (23).

Finally, we show that Ω_ψ is lower semicontinuous on $\mathcal{K}_0^n$. The proof complements the proofs in [22] and [30]. Let $K \in \mathcal{K}_0^n$ and $\varepsilon > 0$ be chosen. Since $\kappa_0(K,\cdot)$ is measurable a.e. on ∂K and since the set ω_0, where the singular part of $C(K,\cdot)$ is concentrated, is a μ_K null set, we can choose by Lusin's theorem (see, for example, [9]) pairwise disjoint closed sets $\omega_l \subset \partial K$, $l \in \mathbb{N}$, such that $\kappa_0(K,\cdot)$ is continuous as a function restricted to ω_l, such that for every $l \in \mathbb{N}$,

$$\omega_l \cap \omega_0 = \emptyset \tag{24}$$

and such that

$$\mu_K\left(\bigcup_{l=1}^{\infty} \omega_l\right) = \mu_K(\partial K). \tag{25}$$

For $\omega \subset \mathbb{R}^n$, let $\bar{\omega}$ be the cone generated by ω, i.e., $\bar{\omega} = \{tx \in \mathbb{R}^n\colon\ t \geqslant 0,\ x \in \omega\}$. Note that $\bar{\omega}_l$ is closed and that $\partial K \cap \bar{\omega}_l = \omega_l$.

Let K_j be a sequence of convex bodies converging to K. First, we show that for $l \in \mathbb{N}$,

$$\liminf_{j\to\infty} \int_{\partial K_j \cap \bar{\omega}_l} \psi\big(\kappa_0(K_j,x)\big)\, d\mu_{K_j}(x) \geqslant \int_{\partial K \cap \bar{\omega}_l} \psi\big(\kappa_0(K,x)\big)\, d\mu_K(x). \tag{26}$$

Let $\eta > 0$ be chosen. We choose a monotone sequence $t_i \in (0,\infty)$, $i = \mathbb{Z}$, $\lim_{i\to-\infty} t_i = 0$, $\lim_{i\to\infty} t_i = \infty$, such that

$$\max_{i\in\mathbb{Z}} \big|\psi(t_{i+1}) - \psi(t_i)\big| \leqslant \eta \tag{27}$$

and such that for $i \in \mathbb{Z}$, $j \geqslant 0$,

$$\mu_{K_j}\big(\big\{x \in \partial K_j\colon\ \kappa_0(K_j,x) = t_i\big\}\big) = 0, \tag{28}$$

where $K_0 = K$. This is possible, since $\mu_{K_j}(\{x \in K_j\colon\ \kappa_0(K_j,x) = t\}) > 0$ holds only for countably many t. Set

$$\omega_{li} = \big\{x \in \omega_l\colon\ t_i \leqslant \kappa_0(K,x) \leqslant t_{i+1}\big\}.$$

Since $\kappa_0(K,\cdot)$ is continuous on ω_l and ω_l is closed, the sets $\bar{\omega}_{li}$ are closed for $i \in \mathbb{Z}$. This implies by (11) that

$$\limsup_{j\to\infty} C(K_j,\bar{\omega}_{li}) \leqslant C(K,\bar{\omega}_{li}). \tag{29}$$

By (24), (15), and the definition of ω_{li},

$$C(K,\bar{\omega}_{li}) = C^a(K,\bar{\omega}_{li}) \leqslant t_{i+1}\mu_K(\partial K \cap \bar{\omega}_{li}). \tag{30}$$

By (16),

$$\int_{\partial K_j \cap \bar{\omega}_{li}} \kappa_0(K_j,x)\, d\mu_{K_j}(x) \leqslant C(K_j,\bar{\omega}_{li}). \tag{31}$$

Using the monotonicity of ψ, we obtain

$$\int_{\omega_l} \psi\big(\kappa_0(K,x)\big)\,d\mu_K(x) \leqslant \sum_{i\in\mathbb{Z}} \int_{\omega_{li}} \psi\big(\kappa_0(K,x)\big)\,d\mu_K(x)$$
$$\leqslant \sum_{i\in\mathbb{Z}} \psi(t_i)\mu_K(\omega_{li}). \tag{32}$$

Using (28), the Jensen inequality, (31), and the monotonicity of ψ, we obtain

$$\begin{aligned}\int_{\partial K_j\cap\bar\omega_l} \psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x) &= \sum_{i\in\mathbb{Z}} \int_{\partial K_j\cap\bar\omega_{li}} \psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x)\\ &= {\sum_{i\in\mathbb{Z}}}' \int_{\partial K_j\cap\bar\omega_{li}} \psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x)\\ &\geqslant {\sum_{i\in\mathbb{Z}}}' \psi\left(\frac{C(K_j,\bar\omega_{li})}{\mu_{K_j}(\partial K_j\cap\bar\omega_{li})}\right)\mu_{K_j}(\partial K_j\cap\bar\omega_{li}),\end{aligned}$$

where the $'$ indicates that we sum only over $\bar\omega_{li}$ with $\mu_{K_j}(\partial K_j\cap\bar\omega_{li})\neq 0$. Since

$$\begin{aligned}&\liminf_{j\to\infty} {\sum_{i\in\mathbb{Z}}}' \psi\left(\frac{C(K_j,\bar\omega_{li})}{\mu_{K_j}(\partial K_j\cap\bar\omega_{li})}\right)\mu_{K_j}(\partial K_j\cap\bar\omega_{li})\\ &\quad\geqslant {\sum_{i\in\mathbb{Z}}}' \psi\left(\limsup_{j\to\infty}\left(\frac{C(K_j,\bar\omega_{li})}{\mu_{K_j}(\partial K_j\cap\bar\omega_{li})}\right)\right)\liminf_{j\to\infty}\mu_{K_j}(\partial K_j\cap\bar\omega_{li}),\end{aligned}$$

we obtain by (29), (30), (32), (27), and (28) that

$$\begin{aligned}&\liminf_{j\to\infty} \int_{\partial K_j\cap\bar\omega_l} \psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x)\\ &\quad\geqslant {\sum_{i\in\mathbb{Z}}}' \psi\left(\frac{C(K,\bar\omega_{li})}{\mu_K(\partial K\cap\bar\omega_{li})}\right)\mu_K(\partial K\cap\bar\omega_{li})\\ &\quad\geqslant \sum_{i\in\mathbb{Z}} \psi(t_{i+1})\mu_K(\partial K\cap\bar\omega_{li})\\ &\quad= \sum_{i\in\mathbb{Z}} \psi(t_i)\mu_K(\partial K\cap\bar\omega_{li}) - \sum_{i\in\mathbb{Z}}\big(\psi(t_i)-\psi(t_{i+1})\big)\mu_K(\partial K\cap\bar\omega_{li})\\ &\quad\geqslant \int_{\partial K\cap\bar\omega_l} \psi\big(\kappa_0(K,x)\big)\,d\mu_K(x) - \eta\mu_K(\partial K\cap\bar\omega_l).\end{aligned}$$

Since $\eta>0$ is arbitrary, this proves (26).

Finally, (28) and (26) imply

$$\begin{aligned}\liminf_{j\to\infty}\int\limits_{\partial K_j}\psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x) &= \liminf_{j\to\infty}\sum_{l=1}^{\infty}\int\limits_{\partial K_j\cap\bar\omega_l}\psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x)\\ &\geqslant \sum_{l=1}^{\infty}\liminf_{j\to\infty}\int\limits_{\partial K_j\cap\bar\omega_l}\psi\big(\kappa_0(K_j,x)\big)\,d\mu_{K_j}(x)\\ &\geqslant \int\limits_{\partial K}\psi\big(\kappa_0(K,x)\big)\,d\mu_K(x).\end{aligned}$$

This completes the proof of the theorem.

6. Open problems

The affine surface areas Ω_ψ and Ω_ψ^* for $\psi \in \mathrm{Conv}(0,\infty)$ are lower semicontinuous and $\mathrm{SL}(n)$ invariant valuations. More general examples of such functionals are

$$\Psi = \Omega_{\psi_1} + \Omega^*_{\psi_2} - \Omega_\phi$$

for $\psi_1, \psi_2 \in \mathrm{Conv}(0,\infty)$ and $\phi \in \mathrm{Conc}(0,\infty)$. Additional examples are the continuous functionals

$$K \mapsto c_0 + c_1|K| + c_2|K^*|$$

for $c_0, c_1, c_2 \in \mathbb{R}$. In view of Theorem 2, this gives rise to the following

Conjecture 1. *If $\Psi : \mathcal{K}_0^n \to (-\infty,\infty]$ is a lower semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation, then there exist $\psi_1, \psi_2 \in \mathrm{Conv}(0,\infty)$, $\phi \in \mathrm{Conc}(0,\infty)$, and $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$\Psi(K) = c_0 + c_1|K| + c_2|K^*| + \Omega_{\psi_1}(K) + \Omega^*_{\psi_2}(K) - \Omega_\phi(K)$$

for every $K \in \mathcal{K}_0^n$.

The following special case of the above conjecture is of particular interest.

Conjecture 2. *If $\Psi : \mathcal{K}_0^n \to (-\infty,\infty]$ is a lower semicontinuous and* $\mathrm{SL}(n)$ *invariant valuation that is homogeneous of degree $q < -n$ or $q > n$, then there exists $c \geqslant 0$ such that*

$$\Psi(K) = c\,\Omega_p(K)$$

for every $K \in \mathcal{K}_0^n$, where $p = n(n-q)/(n+q)$.

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